Universal Outage Behavior of Randomly Precoded Integer Forcing Over MIMO Channels

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Abstract—**Integer forcing is an equalization scheme for the multiple-input multiple-output communication channel that is applicable when all data streams are encoded using a common linear code. The scheme has been demonstrated to allow operating close to capacity for "most" channel matrices. In this work, the measure of "bad" channels is quantified by considering the outage probability of integer forcing, where random unitary precoding is applied at the transmitter side, and where the transmitter only knows the mutual information of the channel.**

I. Introduction

The Multiple-Input Multiple-Output (MIMO) Gaussian channel is central to modern communication and has been extensively studied over the past several decades. Nonetheless, while the capacity limits, under different assumptions on the availability of channel state information, are well understood, the design of low-complexity communication schemes that approach these limits still poses challenges in some scenarios.

For a static channel and a single-user closed-loop setting, capacity may be approached at reasonable complexity. For instance, one may use the singular-value decomposition (SVD) to transform the channel into parallel scalar additive white Gaussian noise (AWGN) channels, over which standard codes may be employed. Alternatively, standard scalar codes may be used in conjunction with the QR matrix decomposition and successive interference cancelation (SIC) decoding, see, e.g., [1]. Coding for MIMO channels in an ergodic fading environment is more involved but has also been successfully addressed. See, e.g., [2].

In contrast, the focus of this paper is on static MIMO channels where the transmitter only knows (or may only utilize its knowledge of) the mutual information of the channel. More specifically, we address the problem of coding over a compound MIMO channel, as is the case in multicast communication.

The design of a practical coding scheme for a compound MIMO channel using an architecture employing space-time linear processing at the transmitter side and integer-forcing (IF) equalization at the receiver side was proposed in [3]. It was shown that such an architecture *universally* achieves the MIMO capacity up to a constant gap, provided that the space-time precoding satisfies the non-vanishing determinant (NVD) criterion. The derived gap, however, is large and thus is of limited practical value.

In the present work, we retain the general architecture of [3], but study its performance when random unitary precoding is applied over the spatial dimension only. Rather than aiming at guaranteeing successful transmission, we study the outage probability of the scheme. Applying random precoding converts the static channel to an effective stochastic one. Further, drawing the precoding matrix from the isotropic unitary ensemble ensures that all channels having the same mutual information for isotropic signaling, will have the same outage probability; see, e.g., [4].

The outage probability in the considered setting thus corresponds to a "scheme outage". Namely, it is the probability that a random precoding matrix results in an effective channel for which the rate achievable with an IF receiver is lower than the target rate. In order to provide universal performance guarantees, we study the worst-case outage probability w.r.t. all possible singular value combinations corresponding to a given mutual information. Thus, the performance guarantee does not depend on channel statistics.

II. Channel Model and Problem Formulation

The single-user (complex) MIMO channel is described by[1]

$$\boldsymbol{y}_c = \mathbf{H}_c\boldsymbol{x}_c + \boldsymbol{z}_c, \tag{1}$$

where $\mathbf{x}_c \in \mathbb{C}^{N_t}$ is the channel input vector, $\mathbf{y}_c \in \mathbb{C}^{N_r}$ is the channel output vector, $\mathbf{H}_c$ is an $N_r \times N_t$ complex channel matrix, and $\boldsymbol{z}_c$ is an additive noise vector of i.i.d. unit variance circularly symmetric complex Gaussian random variables. The input vector $\boldsymbol{x}_c$ is subject to the power constraint[2]

$$\mathbb{E}(\boldsymbol{x}_c^H\boldsymbol{x}_c) \le N_t \cdot \mathsf{SNR}.$$

We assume that the channel is fixed throughout the whole transmission period.

The mutual information of the channel (1) is maximized by a Gaussian input [5] with covariance matrix $\mathbf{Q}$ satisfying $\mathrm{Tr}(\mathbf{Q}) = N_t\mathsf{SNR}$, and is given by

$$C = \max_{\mathbf{Q}:\mathrm{Tr}(\mathbf{Q})\le N_t\mathsf{SNR}} \log\det\left(\mathbf{I}_{N_r\times N_r} + \mathbf{H}_c\mathbf{Q}\mathbf{H}_c^H\right). \tag{2}$$

[1]We denote all complex variables with c to distinguish them from their real-valued representation.

[2]We denote by $[\cdot]^T$, the transpose of a vector/matrix and by $[\cdot]^H$, the Hermitian transpose.

For ease of notation, in the sequel we set $\mathsf{SNR} = 1$, i.e., we "absorb" the SNR into the channel matrix. Thus, we may impose the constraint $\mathrm{Tr}(\mathbf{Q}) \leq N_t$ and replace $\mathbf{H}_c$ in (2) with $\bar{\mathbf{H}}_c = \mathbf{H}_c\sqrt{\mathsf{SNR}}$ and (with abuse of notation) we omit the bar. The choice of $\mathbf{Q}$ that maximizes (2) is determined by the water-filling solution. When the matrix $\mathbf{H}_c$ is known at both transmission ends, i.e., in a closed-loop scenario, this mutual information is the capacity of the channel.

It will prove useful to consider the mutual information achievable with a "white" input. Specifically, taking $\mathbf{Q} = \mathbf{I}_{N_t \times N_t}$, the white-input (WI) mutual information is given by

$$C_{\mathrm{WI}} = \log\det\left(\mathbf{I} + \mathbf{H}_c\mathbf{H}_c^H\right). \tag{3}$$

We may define the set

$$\mathbb{H}(C_{\mathrm{WI}}) = \left\{\mathbf{H}_c \in \mathbb{C}^{N_r \times N_t} : \log\det\left(\mathbf{I} + \mathbf{H}_c\mathbf{H}_c^H\right) = C_{\mathrm{WI}}\right\},$$

of all channel matrices with the same WI mutual information C_{WI}.

The corresponding compound channel model is defined by (1) with the channel matrix $\mathbf{H}_c$ arbitrarily chosen from the set $\mathbb{H}(C_{\mathrm{WI}})$. The matrix $\mathbf{H}_c$ that was chosen by nature is revealed to the receiver, but not to the transmitter. Clearly, the capacity of this compound channel is C_{WI}, and is achieved with a white Gaussian input.

Employing the IF receiver allows operating "close" to C_{WI} for "most" but not all matrices $\mathbf{H}_c \in \mathbb{H}(C_{\mathrm{WI}})$. We quantify the measure of the bad channel matrices by considering outage events. To that end, denote the achievable rate for a given channel matrix $\mathbf{H}_c$ with an IF receiver as $R_{\mathrm{IF}}(\mathbf{H}_c)$. An explicit expression for this rate is given in Section III-A. Since applying a precoding matrix $\mathbf{P}$ results in an effective channel $\mathbf{H}_c \cdot \mathbf{P}$, it follows that the achievable rate of IF for this channel is $R_{\mathrm{IF}}(\mathbf{H}_c \cdot \mathbf{P})$. Therefore, the worst-case (WC) scheme outage is defined as

$$P_{\mathrm{out}}^{\mathrm{WC}}(C_{\mathrm{WI}}, R) = \sup_{\mathbf{H}_c \in \mathbb{H}(C_{\mathrm{WI}})} \Pr\left(R_{\mathrm{IF}}(\mathbf{H}_c \cdot \mathbf{P}) < R\right), \tag{4}$$

where the probability is over the ensemble of precoding matrices. The goal of this paper is to quantify the tradeoff between the transmission rate R and the outage probability of IF as defined in (4).

III. Integer-forcing: background

A. Single-User Integer-Forcing Equalization

In [6], a receiver architecture scheme coined "integer forcing" was proposed. We recall the achievable rate of this scheme.

We follow the derivation of [6] and describe integer forcing over the reals. Channel model (1) can be expressed via its real-valued representation as

$$\underbrace{\begin{bmatrix}\mathrm{Re}(\mathbf{y}_c)\\ \mathrm{Im}(\mathbf{y}_c)\end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix}\mathrm{Re}(\mathbf{H}_c) & -\mathrm{Im}(\mathbf{H}_c)\\ \mathrm{Im}(\mathbf{H}_c) & \mathrm{Re}(\mathbf{H}_c)\end{bmatrix}}_{\mathbf{H}} \underbrace{\begin{bmatrix}\mathrm{Re}(\mathbf{x}_c)\\ \mathrm{Im}(\mathbf{x}_c)\end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix}\mathrm{Re}(\mathbf{z}_c)\\ \mathrm{Im}(\mathbf{z}_c)\end{bmatrix}}_{\mathbf{z}}.$$

This real-valued representation is used in the sequel to derive performance bounds for the complex channel $\mathbf{H}_c$. Note that the dimensions of $\mathbf{H}$ are $2N_r \times 2N_t$.

Any real MIMO channel can be described via its singular-value decomposition (SVD) $\mathbf{H} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$. The following eigenvalue decomposition can thus be easily derived

$$(\mathbf{I} + \mathbf{H}^T\mathbf{H})^{-1} = \mathbf{V}\mathbf{D}^{-1}\mathbf{V}^T, \tag{5}$$

where $\mathbf{D} = \mathbf{I} + \mathbf{\Sigma}^2$. It is shown in [6] that for a given integer coefficient vector $\mathbf{a}_m$, the maximal possible rate for decoding the associated linear combination of messages is

$$R_{\mathrm{IF}}^{\mathbf{a}_m}(\mathbf{H}) = R_{\mathrm{IF}}^{\mathbf{a}_m}(\mathbf{D}, \mathbf{V}) = -\frac{1}{2}\log\left(\|\mathbf{D}^{-1}\mathbf{V}^T\mathbf{a}_m\|^2\right). \tag{6}$$

By Theorem 3 in [6], transmission with IF equalization can achieve any rate satisfying $R < R_{\mathrm{IF}}(\mathbf{H})$ where

$$R_{\mathrm{IF}}(\mathbf{H}) = -N_t \log\left(\min_{\substack{\mathbf{A} \in \mathbb{Z}^{2N_t \times 2N_t}\\ \det\mathbf{A} \neq 0}} \max_m \boldsymbol{a}_m^T(\mathbf{I} + \mathbf{H}^T\mathbf{H})^{-1}\boldsymbol{a}_m\right),$$

and where $\mathbf{A} = [\boldsymbol{a}_1 \cdots \boldsymbol{a}_{N_t}]^T$.

Let Λ be the lattice spanned by $\mathbf{G} = \mathbf{D}^{-1/2}\mathbf{V}^T$. The achievable rates of IF may also be described via the successive minima of this lattice. Recall the definition of successive minima.

Definition 1. *Let $\Lambda(\mathbf{G})$ be the lattice spanned by the full-rank matrix $\mathbf{G} \in \mathbb{R}^{K \times K}$. For $k = 1, ..., K$, we define the k'th successive minimum as*

$$\lambda_k(\mathbf{G}) \triangleq \inf\left\{r : \dim\left(\mathrm{span}\left(\Lambda(\mathbf{G}) \cap \mathcal{B}(\mathbf{0}, r)\right)\right) \geq k\right\} \tag{7}$$

where $\mathcal{B}(\mathbf{0}, r) = \left\{\mathbf{x} \in \mathbb{R}^K : \|\mathbf{x}\| \leq r\right\}$ is the closed ball of radius r around $\mathbf{0}$. In words, the k'th successive minimum of a lattice is the minimal radius of a ball centered around $\mathbf{0}$ that contains k linearly independent lattice points.

Thus, we have

$$R_{\mathrm{IF}}(\mathbf{D}, \mathbf{V}) = -2N_t\frac{1}{2}\log\left(\lambda_{2N_t}^2(\Lambda)\right) = N_t\log\left(\frac{1}{\lambda_{2N_t}^2(\Lambda)}\right). \tag{8}$$

B. Precoded Integer-Forcing

The transmission scheme considered consists of applying unitary precoding at the transmitter and IF equalization at the receiver, and will be referred to as precoded integer-forcing (P-IF). Precoding may be viewed as generating a "virtual" channel $\widetilde{\mathbf{H}}_c = \mathbf{H}_c\mathbf{P}$, over which transmission takes place.

Throughout this paper, we assume that the precoding matrix $\mathbf{P} = \widetilde{\mathbf{V}}_c$ is drawn from what is referred to as the "circular unitary ensemble" (CUE). The ensemble is defined by the unique distribution on unitary matrices that is invariant under left and right unitary transformations [7]. That is, given a random matrix $\widetilde{\mathbf{V}}_c$ drawn from the CUE, for any unitary matrix $\mathbf{V}_c$, both $\widetilde{\mathbf{V}}_c\mathbf{V}_c$ and $\mathbf{V}_c\widetilde{\mathbf{V}}_c$ are equal in distribution to $\widetilde{\mathbf{V}}_c$.

The SVD of the precoded channel is $\mathbf{H}_c\widetilde{\mathbf{V}}_c = \mathbf{U}_c\mathbf{\Sigma}_c\mathbf{V}_c^H\widetilde{\mathbf{V}}_c$. Since $\mathbf{V}_c^H\widetilde{\mathbf{V}}_c$ is equal in distribution to $\widetilde{\mathbf{V}}_c$, for the sake of computing outage probabilities, we may simply assume that $\mathbf{V}_c^H$ (and also $\mathbf{V}_c$) is drawn from the CUE.[3]

We may apply the decomposition (5) to the real-valued representation of $\mathbf{H}_c$ to obtain $\mathbf{V}$ and $\mathbf{D}$. It can be shown (see [8]) that the rates of IF, with or without SIC, of such a channel come in pairs. Hence, the entries of $\mathbf{D}$ come in pairs.

The following lemma is simple to derive (and can be found in Appendix A of [8]) and will prove useful in characterizing the performance of P-IF. It expresses the outage probability of IF for precoding with the CUE in terms of that arising when precoding is performed using the real circular orthogonal ensemble (COE).[4]

Lemma 1. *Let $\mathbf{O}$ be a real $2N_t \times 2N_t$ matrix drawn from the COE. When applying a random complex precoding matrix $\mathbf{V}_c$ which is drawn from the CUE (inducing a real-valued precoding matrix $\mathbf{V}$), the following holds*

$$\Pr\left(\left\|\mathbf{D}^{1/2}\mathbf{V}\mathbf{a}\right\| < \sqrt{\beta}\right) = \Pr\left(\left\|\mathbf{D}^{1/2}\mathbf{O}\mathbf{a}\right\| < \sqrt{\beta}\right). \quad (9)$$

IV. Bound on the outage probability of precoded Integer-Forcing

Define the dual lattice Λ^* which is spanned by the matrix $(\mathbf{G}^T)^{-1} = \mathbf{D}^{1/2}\mathbf{V}^T$. Recall that the rate of IF is given by (8). Now, the successive minima of Λ and Λ^* are related by (Theorem 2.4 in [9])

$$\lambda_1(\Lambda^*)^2 \lambda_{2N_t}(\Lambda)^2 \le \frac{2N_t+3}{4} {\gamma_{2N_t}^*}^2, \quad (10)$$

where γ_n is Hermite's constant. Therefore, we may express the achievable rates of IF via the dual lattice as follows

$$R_{\text{IF}}(\mathbf{D},\mathbf{V}) > N_t \log\left(\frac{\lambda_1^2(\Lambda^*)}{\frac{2N_t+3}{4}{\gamma_{2N_t}^*}^2}\right). \quad (11)$$

Hermite's constant is known only for dimensions $1-8$ and 24. Since it has been never proved that γ_{2N_t} is monotonically increasing, we define $\gamma_{2N_t}^* = \max\{\gamma_i : 1 \le i \le 2N_t\}$.

The tightest known bound for Hermite's constant as derived in [10] is

$$\gamma_{2N_t} \le (2/\pi)\cdot\Gamma\left(2+N_t\right)^{1/N_t}. \quad (12)$$

Since this is an increasing function of N_t, it follows that $\gamma_{2N_t}^*$ is smaller than the r.h.s. of (12). This implies that

$$R_{\text{IF}}(\mathbf{D},\mathbf{V}) \ge N_t \log\left(\frac{\lambda_1^2(\Lambda^*)}{\alpha(N_t)}\right),$$

where

$$\alpha(N_t) = \begin{cases} \frac{2N_t+3}{4}\gamma_{2N_t}^2, & N_t = 2,3,4,12 \\ \frac{2N_t+3}{4}\left(\frac{2}{\pi}\Gamma\left(2+N_t\right)^{1/N_t}\right)^2, & \text{otherwise} \end{cases}. \quad (13)$$

For simplicity of notation, we henceforth denote $C = C_{\text{WI}}$. Defining $\Delta C = C - R$, we have

$$\begin{aligned}&\Pr\left(R_{\text{IF}}(\mathbf{D},\mathbf{V}) < C - \Delta C\right)\\ &\le \Pr\left(N_t \log\left(\frac{\lambda_1^2(\Lambda^*)}{\alpha(N_t)}\right) < C - \Delta C\right)\\ &= \Pr\left(\lambda_1^2(\Lambda^*) < 2^{\frac{C-\Delta C}{N_t}}\alpha(N_t)\right).\end{aligned}$$

We may now rewrite (4) as

$$P_{\text{out}}^{\text{WC}}(C,\Delta C) = \sup_{\mathbf{D}\in\mathbb{D}(C)} \Pr\left(R_{\text{IF}}(\mathbf{D},\mathbf{V}) < C - \Delta C\right)$$

where $\mathbb{D}(C)$ is the set of all diagonal matrices $\mathbf{D}$, with diagonal elements coming in pairs, such that $\det(\mathbf{D}) = 2^C$.

Lemma 2. *For any $N_r \times N_t$ complex channel with WI mutual information C, i.e., $\mathbf{D} \in \mathbb{D}(C)$, and for $\mathbf{V}_c$ drawn from the CUE (inducing a real-valued precoding matrix $\mathbf{V}$), we have*

$$\Pr\left(R_{\text{IF}}(\mathbf{D},\mathbf{V}) < C - \Delta C\right) \le \sum_{\mathbf{a}\in\mathbb{A}(\beta,d_{\min})} \frac{2N_t\left(2^{\frac{C-\Delta C}{N_t}}\alpha(N_t)\right)^{N_t-1/2}}{\|\mathbf{a}\|^{2N_t-1} 2^{C\frac{2}{\sqrt{d_{\min}}}}},$$

where

$$\mathbb{A}(\beta,d_{\min}) = \left\{\mathbf{a} : 0 < \|\mathbf{a}\| < \sqrt{\frac{\beta}{d_{\min}}}\right\},$$

with $\beta = 2^{\frac{C-\Delta C}{N_t}}\alpha(N_t)$ and $d_{\min} = \min_i \mathbf{D}_{i,i}$.

Proof. For some $\beta > 0$, let us upper bound the probability $\Pr\left(\lambda_1^2(\Lambda^*) < \beta\right) = \Pr\left(\lambda_1(\Lambda^*) < \sqrt{\beta}\right)$. Noting that the event $\left\{\lambda_1(\Lambda^*) < \sqrt{\beta}\right\}$ is equivalent to the event

$$\bigcup_{\mathbf{a}\in\mathbb{Z}^{2N_t}\setminus\{\mathbf{0}\}} \left\{||\mathbf{D}^{1/2}\mathbf{V}\mathbf{a}|| < \sqrt{\beta}\right\}$$

and applying the union bound gives

$$\begin{aligned}\Pr\left(\lambda_1(\Lambda^*) < \sqrt{\beta}\right) &< \sum_{\mathbf{a}\in\mathbb{Z}^{2N_t}\setminus\{\mathbf{0}\}} \Pr\left(\left\|\mathbf{D}^{1/2}\mathbf{V}\mathbf{a}\right\| < \sqrt{\beta}\right)\\ &= \sum_{\mathbb{A}(\beta,d_{\min})} \Pr\left(\|\mathbf{D}^{1/2}\mathbf{V}\mathbf{a}\| < \sqrt{\beta}\right),\end{aligned} \quad (14)$$

where the second inequality follows since whenever $\|\mathbf{a}\|\cdot\sqrt{d_{\min}} \ge \sqrt{\beta}$, we have $\Pr\left(\|\mathbf{D}^{1/2}\mathbf{V}\mathbf{a}\| < \sqrt{\beta}\right) = 0$.

Let $\mathcal{S}$ denote the unit sphere and denote $\mathbf{o}_{\|\mathbf{a}\|} \sim \text{Unif}(\mathcal{S}\cdot\|\mathbf{a}\|)$. Recalling Lemma 1 and noting

[3] In case $\mathbf{H}_c$ is drawn from an ensemble for which $\mathbf{V}_c$ is uniformly distributed (equivalently, is drawn from the CUE), the precoding operation is redundant.

[4] The COE is defined analogously to the CUE, for the case of real orthonormal matrices [7].

that $\mathbf{Oa}$ is equal in distribution to $\mathbf{o}_{\|\mathbf{a}\|}$, it follows that

$$\Pr\left(\|\mathbf{D}^{1/2}\mathbf{Va}\| < \sqrt{\beta}\right) = \Pr\left(\|\mathbf{D}^{1/2}\mathbf{o}_{\|\mathbf{a}\|}\| < \sqrt{\beta}\right). \quad (15)$$

Now the probability appearing on the r.h.s. of (15) has a simple geometric interpretation. Define an ellipsoid with axes $x_i = \sqrt{d_i} \cdot \|\mathbf{a}\|$ and denote its surface area by $L(x_1, x_2, ..., x_{2N_t})$. Then, the r.h.s. of (15) is the ratio of the part of the surface area of an ellipsoid which is contained inside a sphere of radius $\sqrt{\beta}$ (denoted by $\mathrm{CAP_{ell}}(x_1, x_2, ..., x_{2N_t})$) and the total surface area of the ellipsoid. This is illustrated in Figure 1 for the case of two real dimensions. We may rewrite (15) as

$$\Pr\left(\|\mathbf{D}^{1/2}\mathbf{o}_{\|\mathbf{a}\|}\| < \sqrt{\beta}\right) = \frac{\mathrm{CAP_{ell}}(a_1, a_2, ..., a_{2N_t})}{L(a_1, a_2, ..., a_{2N_t})} \quad (16)$$

$$= \frac{|\mathbf{D}^{1/2}\mathcal{S} \cdot \|\mathbf{a}\| \cap \sqrt{\beta}\mathcal{S}|}{|\mathbf{D}^{1/2}\mathcal{S}\|\mathbf{a}\||}. \quad (17)$$

Neither the numerator nor the denominator of (17) has a closed-form expression. In order to upper bound this ratio, we upper bound the numerator and lower bound the denominator.

Using (54) and (57) in [11], we have

$$L(x_1, x_2, ..., x_{2N_t}) > \Omega_{2N_t}\|\mathbf{a}\|^{2N_t} \prod_{i=1}^{2N_t} \sqrt{d_i} \sum_{i=1}^{2N_t} \frac{1}{\|\mathbf{a}\|\sqrt{d_i}}, \quad (18)$$

where $\Omega_{2N_t} = \frac{\pi^{N_t}}{\Gamma(1+N_t)}$ is the volume of a unit ball of dimension $2N_t$.

As an upper bound for the numerator, we take the entire surface area of the sphere with radius $\sqrt{\beta}$. Recalling that the surface area of a sphere of radius $\sqrt{\beta}$ is $A_{2N_t}\left(\sqrt{\beta}\right) = 2N_t \frac{\pi^{N_t}}{\Gamma(1+N_t)} \sqrt{\beta}^{2N_t-1}$, we thus have

$$\mathrm{CAP_{ell}}(\mathrm{x}_1, \mathrm{x}_2, ..., \mathrm{x}_{2\mathrm{N}_t}) \le A_{2N_t}\left(\sqrt{\beta}\right). \quad (19)$$

Substituting (18), (19) into (17) yields

$$\sum_{\mathbb{A}(\beta, d_{\min})} \Pr\left(\|\mathbf{D}^{1/2}\mathbf{o}_{\|\mathbf{a}\|}\| < \sqrt{\beta}\right) <$$

$$\sum_{\mathbb{A}(\beta, d_{\min})} \frac{2N_t \frac{\pi^{N_t}}{\Gamma(1+N_t)} \sqrt{\beta}^{2N_t-1}}{\frac{\pi^{N_t}}{\Gamma(1+N_t)} \|\mathbf{a}\|^{2N_t-1} 2^{C} \frac{2}{\sqrt{d_{\min}}}}.$$

Further substituting $\beta = 2^{\frac{C-\Delta C}{N_t}} \cdot \alpha(N_t)$, we finally arrive at

$$\Pr\left(R_{\mathrm{IF}}(\mathbf{D}, \mathbf{V}) < C - \Delta C\right)$$

$$\le \sum_{\mathbb{A}(\beta, d_{\min})} \frac{2N_t \left(2^{\frac{C-\Delta C}{N_t}} \alpha(N_t)\right)^{N_t - 1/2}}{\|\mathbf{a}\|^{2N_t-1} 2^{C} \frac{2}{\sqrt{d_{\min}}}}. \quad (20)$$

□

The bound of Lemma 2 is depicted in Figure 2. Rather than plotting the outage probability, its complement is depicted, i.e., we depict the CDF. For given C and ΔC, Lemma 2

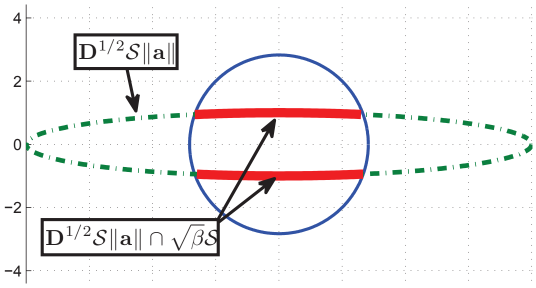

Fig. 1. Illustration of the geometric objects appearing in (17-19).

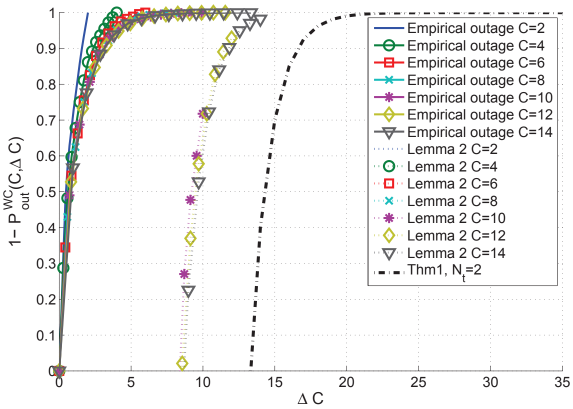

Fig. 2. Comparing Lemma 2 against empirical results for $N_r \times 2$ complex channels for various values of capacity.

was numerically calculated over a grid of singular values. For each such vector of singular values.the summation of (20) is computed. The worst-case outage probability over all vectors of singular values from the grid is presented. In addition, empirical results are also plotted, where for each vector of singular values, a large number of random unitary matrices was drawn and the outage probability was calculated. Then, the worst-case outage probability over all tested singular values (i.e., those belonging to the grid) is presented.

Lemma 2 provides an explicit bound on the outage probability. Nonetheless, in order to calculate it, one needs to go over all diagonal matrices in $\mathbb{D}(C)$ and for each diagonal matrix, sum over all the relevant integer vectors in $\mathbb{A}(\beta, d_{\min})$. The following theorem provides a simpler bound, also depicted in Figure 2, that does not depend on capacity and hence can be evaluated analytically for any number of antennas. The proof appears in Appendix B of [8].

Theorem 1. *For any $N_r \times N_t$ complex channel with WI mutual information C, and for $\mathbf{V}_c$ drawn from the CUE (which induces a real-valued precoding matrix $\mathbf{V}$), we have*

$$P_{\mathrm{out}}^{\mathrm{WC}}(C, \Delta C) \le c(N_t) 2^{-\Delta C},$$

where $c(N_t)$ is a constant that depends only on N_t.

V. Application : Universal Gap-To-Capacity for Multi-User Closed-Loop MultiCast using P-IF

In closed-loop MIMO multicast, a transmitter equipped with N_t transmit antennas wishes to send the same message to K users, where user i is equipped with N_i antennas.

Even though channel state information is available at both transmission ends, designing practical capacity-approaching schemes for closed-loop multicast with $K \geq 3$ users is challenging as detailed in [12]. Specifically, to achieve a small gap to capacity, the scheme of [12] requires utilizing space-time coding with a large number of channel uses. The outage bound derived above suggests that P-IF may be an attractive practical closed-loop MIMO multicast scheme, allowing to obtain a small gap to capacity with space-only precoding.

Denoting by $\mathbf{H}_{c,i}$ the $N_i \times N_t$ channel matrix corresponding to the i'th user and by $\mathcal{H} = \{\mathbf{H}_{c,i}\}_{i=1}^{K}$ the set of channels, the received signal at user i is

$$\boldsymbol{y}_{c,i} = \mathbf{H}_{c,i}\boldsymbol{x}_c + \boldsymbol{z}_{c,i}. \tag{21}$$

The multicast capacity is defined as the capacity of the compound channel (21). It is attained by a Gaussian vector input, where the mutual information is maximized over all covariance matrices $\mathbf{Q}$ satisfying $\mathrm{Tr}(\mathbf{Q}) \leq N_t$:

$$C(\mathcal{H}) = \max_{\mathbf{Q}:\mathrm{Tr}(\mathbf{Q})\leq N_t} \min_{\mathbf{H}_c \in \mathcal{H}} \log\det(\mathbf{I} + \mathbf{H}_c\mathbf{Q}\mathbf{H}_c^H).$$

We assume without loss of generality that the input covariance matrix is the identity matrix.[5]

We note that for each user i there exists an $\alpha_i \geq 1$ such that $\mathbf{H}_{c,i} = \alpha_i \breve{\mathbf{H}}_{c,i}$, where $\breve{\mathcal{H}} = \{\breve{\mathbf{H}}\}_{i=1}^{K} \in \mathbb{H}(C(\mathcal{H}))$, i.e., $\breve{\mathcal{H}}$ is contained in the (continuum) set of channels with the same capacity $C(\mathcal{H})$. Further, α_i can be interpreted as excess SNR which exists for user i. Since the achievable rate of IF is monotonically increasing in SNR, it follows that the achievable rates over the set of channels $\mathcal{H}$ can only be higher than over $\breve{\mathcal{H}}$, which we next lower bound.

Let us consider applying the random precoded scheme of Section IV to the compound channel set $\breve{\mathcal{H}}$. Define $A_i(R)$ as the event where a matrix $\widetilde{\mathbf{V}}_c$ drawn from the CUE achieves a desired target R for user i

$$A_i(R) = \left\{ \widetilde{\mathbf{V}}_c : R_{\mathrm{IF}}(\mathbf{H}_{c,i} \cdot \widetilde{\mathbf{V}}_c) \geq R \right\}.$$

We're interested in the probability of achieving the target rate for all users, i.e., $\Pr(\cap A_i(R))$. Using the union bound,

$$\Pr(\cap A_i(R)) \geq 1 - K P_{\mathrm{out}}^{\mathrm{WC}}(C(\mathcal{H}), R), \tag{22}$$

where $P_{\mathrm{out}}^{\mathrm{WC}}(\cdot)$ is defined in (4).

This provides a means to obtain a guaranteed achievable closed-loop P-IF transmission rate $R_{\mathrm{WC-CL}}(\mathcal{H})$. Namely, $R_{\mathrm{WC-CL}}(\mathcal{H})$ is the maximum rate R for which

$$P_{\mathrm{out}}^{\mathrm{WC}}(C(\mathcal{H}), R) \leq 1/K. \tag{23}$$

Substituting (23) in (22), we have for any $R < R_{\mathrm{WC-CL}}(\mathcal{H})$,

$$\Pr(\cap A_i(R)) > 1 - K \cdot (1/K) = 0.$$

Thus, there exists a precoding matrix for which any target rate $R < R_{\mathrm{WC-CL}}(\mathcal{H})$ is achievable (via P-IF transmission) for the compound channel (21). For example, in the case of $N_t = 2$, using the tightest upper bound developed thus far (Lemma 2), we get a guaranteed gap-to-capacity of 9.6, 10.2 and 10.5 bits for 2, 3, and 4 users, respectively.

Examining Lemma 2 reveals that there are two major sources for looseness (see [8]). First, some terms in the summation (14) may be dropped. Second, bounding via the dual lattice induces a loss reflected in (10). This may be circumvented for $N_t = 2$ using IF-SIC. Corollary 1 of Lemma 3 in [8] addresses these two issues and for the example above, tightens the gap-to-capacity to 3.24, 3.8 and 4.385 bits for 2, 3, and 4 users, respectively.

To conclude, we used the probabilistic method to obtain a universal guarantee on the gap-to capacity for closed-loop MIMO multicast via P-IF . We note that in [13] it is numerically demonstrated that the gap-to-capacity is in fact much smaller when searching for the *optimal* precoding matrix.

[5]We may do so since the covariance shaping matrix $\mathbf{Q}^{1/2}$ may be absorbed into the channel by defining the effective channel $\hat{\mathbf{H}}_{c,i} = \mathbf{H}_{c,i}\mathbf{Q}^{1/2}$. Thus, $C(\mathcal{H}) = \min_i C_{\mathrm{WI}}(\hat{\mathbf{H}}_{c,i})$. With a slight abuse of notation, we use $\mathbf{H}_{c,i}$ to denote the effective channel.

References

[1] D. Tse and P. Viswanath, *Fundamentals of wireless communication*. Cambridge university press, 2005.

[2] B. Hochwald and S. ten Brink, "Achieving near-capacity on a multiple-antenna channel," *Communications, IEEE Transactions on*, vol. 51, no. 3, pp. 389–399, March 2003.

[3] O. Ordentlich and U. Erez, "Precoded Integer-Forcing Universally Achieves the MIMO Capacity to Within a Constant Gap," *Information Theory, IEEE Transactions on*, vol. 61, no. 1, pp. 323–340, Jan 2015.

[4] E. Larsson, "Constellation randomization (CoRa) for outage performance improvement on MIMO channels," in *Global Telecommunications Conference, 2004*, vol. 1, Nov 2004, pp. 386–390.

[5] I. E. Telatar, "Capacity of multi-antenna Gaussian channels," *Eur. Trans. Telecommun*, vol. 10, pp. 585–598, Nov 1999.

[6] J. Zhan, B. Nazer, U. Erez, and M. Gastpar, "Integer-forcing linear receivers," *Information Theory, IEEE Transactions on*, vol. 60, no. 12, pp. 7661–7685, Dec 2014.

[7] M. Metha, "Random matrices and the statistical theory of energy levels," *Academic, New York*, 1967.

[8] E. Domanovitz and U. Erez, "Outage Behavior of Randomly Precoded Integer Forcing for MIMO Channels." [Online]. Available: https://www.eng.tau.ac.il/~uri/papers/outage_precoded_IF.pdf

[9] J. Lagarias, J. Lenstra, H.W., and C. Schnorr, "Korkin-Zolotarev bases and successive minima of a lattice and its reciprocal lattice," *Combinatorica*, vol. 10, no. 4, pp. 333–348, 1990.

[10] H. Blichfeldt, "The minimum value of quadratic forms, and the closest packing of spheres," *Mathematische Annalen*, vol. 101, no. 1, pp. 605–608, 1929.

[11] G. Tee, "Surface Area and Capacity of Ellipsoids in n Dimensions," CITR, The University of Auckland, New Zealand, Tech. Rep., 2004.

[12] A. Khina, I. Livni, A. Hitron, and U. Erez, "Joint unitary triangularization for Gaussian multi-user MIMO networks," *IEEE Transactions on Information Theory*, vol. 61, no. 5, pp. 2662–2692, May 2015.

[13] E. Domanovitz and U. Erez, "Performance of precoded integer-forcing for closed-loop MIMO multicast," in *ITW 2014*, Nov 2014, pp. 282–286.